11

Utilitarianism versus Fairness in Welfare Economics

Marc Fleurbaey and François Maniquet

11.1 Introduction

Utilitarianism has been opposed to theories of fairness (especially Rawls’s theory) in many respects. We want to focus here on a particular division that has been seldom discussed, although it is reflected in the structure of welfare economics. Welfare economics is indeed currently separated into two very different branches. One branch deals with social welfare functions and devotes a substantial energy to the study of utilitarianism. The other studies fair allocation in economic models and, formally, its main focus is on allocation rules. The difference is the following. A social welfare function associates each member in a class of possible contexts with a ranking of all possible alternatives, whereas an allocation rule only associates each member in the class with a selection of “best” alternatives. As it has long since been noted in the theory of social choice, an allocation rule is a kind of ranking, albeit simple (any two selected allocations as well as any two nonselected allocations being deemed socially indifferent), and a ranking immediately leads to an allocation rule (which selects the best alternative in every context).

Actually, there is a second important difference between these two branches. The arguments of the social welfare functions studied by the former are interpersonally comparable utilities (usually comparable in levels, differences, or ratios), whereas the whole body of literature representing the latter is purely ordinal, making use of no other welfare information than the preferences of the agents over simple alternatives.¹

¹Excellent recent summaries of the two branches are available in Mongin and d’Aspremont (1998) for the former and Thomson (2008) and Moulin and Thomson (1997) for the latter.

We would like to thank a reviewer and the editors of this volume for many valuable comments and suggestions on an earlier version.
These two branches are not separated because of a theoretical opposition on some deep issue. On the contrary, each school would be happy to share some feature with the other one. The fairness theorists regret that they are unable to compare unfair allocations, whereas some specialists in the other branch would be happy to get rid of the burden of interpersonal comparisons of utility, although most of them think that they cannot avoid such comparisons. We think that these dual drawbacks deprive each of these two branches of most of its practical relevance for policy issues. The policy maker is obviously extremely reluctant to engage in interpersonal comparisons of such impalpable objects as utilities, which makes social welfare functions look like wonderful machines that just lack the appropriate fuel. We shall argue below that such reluctance is ethically sound. However, first best efficient allocation rules are irrelevant to the analysis of piecemeal reforms in an imperfect world. The only way in which policy recommendations have been able to appear in this branch is through sophisticated implementation games that are often considered as requiring too much information and intelligence from the agents. Such games, and the corresponding allocation rules, resemble wonderful machines that only go on roads that do not exist.

What we want to do is simply to study purely ordinal social orderings. If we could find nice objects of this sort, then we would be able to compare any pair of allocations without relying on utilities and interpersonal comparisons of utilities. Knowledge of the agents’ preferences would be enough. We do not, however, want to suggest that such tools could be directly applied, and in particular that revelation problems would be easily solved. But we will argue that ordinal social orderings would be extremely useful tools in some contexts.

The definition of purely ordinal social orderings is of course far from being new because it is the goal of Arrovian social choice theory. The almost exclusively negative results that were obtained in this approach merely reinforced the feeling among theorists that interpersonal comparisons of utility were the price to pay for reasonable social orderings. We think that Arrow’s approach was too demanding in two respects.

First, it considered abstract spaces of alternatives and pretended to solve all aggregation problems at the same time without taking into account specific features of economic allocations. On the contrary, we argue that the economic context (private or public goods, distribution or production, goods or bads, etc.) and the particular features of the alternatives may significantly alter the social judgment, and rightly so, since there are ethical principles that are context-dependent, at least in their precise application.\footnote{See Moulin (1990).}
instance, depending on the returns to scale or the rivalry of consumption, the stand-alone principle either says that an agent should not be better off, or worse off, than if she were alone in the economy. Another example is provided by the no veto power requirement. It is legitimately imposed on voting procedures for a large number of people. In the distribution problem of a single commodity whose consumption may lead to satiation, however, applying no veto power when all agents are identical may lead us to give their preferred consumption level to all but one agent, and her worst consumption level to the last one, thereby violating the basic horizontal equity requirement.

The second excessive requisite in Arrow’s approach was the axiom of Independence of Irrelevant Alternatives. This axiom requires the social ranking of two alternatives to depend only on how the agents rank these two alternatives. We will argue below that even if this axiom has nice justifications, there is no reason to impose it at all cost, and we will show how much it must be weakened to open the way to ordinal social orderings. In conclusion, turning to economic environments and weakening the axiom of Independence is the way in which we propose to avoid the Arrovian impossibilities.\(^3\)

Ordinal social orderings have recently been studied by Maniquet (1994) and more specifically by Bossert, Fleurbaey, and Van de gaer (1999), in a quite general framework where the agents’ characteristics may be anything. We focus here on the basic case where agents differ only in their preferences.

The first section in this chapter presents arguments in favor of ordinal social orderings. In Subsection 11.2.1, we argue in favor of ordinalism and against the idea that interpersonal comparisons of utility are unavoidable. In Subsection 11.2.2, we discuss the relative merits of social orderings and allocation rules. The main topic is tackled in Section 11.3, where examples of ordinal social orderings are studied in the restricted framework of division economies. In a first subsection, we introduce properties which can be directly used to select among plausible social ordering functions. A social ordering function associates a social ordering to each economy in a class of admissible economies. In particular, we propose an alternative to Arrow’s Independence of Irrelevant Alternatives axiom that can be combined quite easily with other axioms. In a second and last subsection, we examine how to derive social ordering functions from allocation rules. Our main result is

\(^3\) The theory of social choice in economic environments (reviewed in Donaldson and Weymark, 1988; and Le Breton, 1997) comes close to our project but has mostly retained the axiom of Independence and the negative flavor of results that ensue.
the proof that the three major solutions to the fair division problem (that is, the Fixed Numeraire Egalitarian Equivalent rule, the Pazner–Schmeidler Egalitarian Equivalent rule, and the Equal Income Walrasian rule) can be rationalized by social ordering functions satisfying basic desirable properties (that is, Weak Pareto, Pareto Indifference, and Anonymity) and the independence axiom we introduce in the preceding subsection. Concluding comments are given in Section 11.4.

11.2 Justifying Social Orderings

11.2.1 Ordinalism versus Interpersonal Comparisons of Utility

In the layman’s (possibly a policy maker’s or at least a voter’s) reluctance to try and make interpersonal comparisons of utility in distributive issues beyond the family circle, one can decipher a mixture of two objections: One cannot make such comparisons and one should not. The former is familiar to economists, the latter has been developed by philosophers and in particular by Rawls.

That consumers’ demand behavior reveals ordinal preferences only is well known and implies that interpersonal comparisons require additional information that may not be easily collected. This is the main reason why economists have problems with interpersonal utility comparisons, and we think that this provides sufficient justification for our ordinal approach. But even if utility information could be gathered costlessly, there are ethical arguments against using it. In Rawls’s (1971, 1982), Dworkin’s (1981) and Van Parijs’s (1995) theories of justice, individuals should assume responsibility for their ends, preferences, and satisfaction levels. The idea is that there would be some contradiction in treating individuals as morally autonomous in the formation of their life plans and, at the same time, redistributing resources so as to guarantee that they reach some utility level, given that utility functions are viewed as part of one’s life plans. The general problem of distributive justice is to allocate resources to individuals who will use them as they wish, not to allocate utility. Also, the local problem of distributive justice, which we address in Section 11.3, that is, the division of unproduced commodities, is to allocate goods in an equal way. In either case, a pure utility deficit does not give a valid claim against others.

This argument is quite convincing because it essentially makes explicit what is currently enacted in liberal societies, where respect for the individual’s private sphere entails such a division of labor between social
institutions and individual initiative. The implications of this argument are far-reaching because they support the idea that the allocation of resources should be independent of individuals’ preferences and life plans, but one must be careful. It would certainly be excessive to propose making the allocation of goods independent of preferences, and the philosophers’ theories do not have such an implication. In Rawls’s theory, only primary goods (which are essentially all-purpose goods) should be distributed independently of preferences, and in Dworkin’s proposal as well as Van Parijs’s proposal, the distribution of income only is discussed, so that in all theories, the market mechanism determines the final allocation of goods, in a way that is obviously dependent on preferences.

Here we follow the usual practice in economic theory, which is to assume nothing about institutions from the outset. We look for orderings of allocations, without assuming the market to have a special role. These arguments, in our opinion, justify that we rely on ordinal noncomparable information only. Moreover, we will retain the idea that the allocation of goods (or the social ordering) should be as independent of preferences as possible but still compatible with the Pareto principle. The idea that the Pareto principle represents the minimal requirement about how much preferences must matter is standard and quite natural because going against unanimous preference seems to be the strongest way in which individuals’ preferences can be disregarded. As a result, the preference independence idea will take the form of independence axioms with respect to changes in preferences.

An often raised objection to our approach must be met before proceeding. This objection could be called the planner’s ethical preferences revelation principle. It is, indeed, often contended that, willy-nilly, interpersonal comparisons of utility are always made in allocation decisions. We strongly oppose this claim. The rule that a cake must be cut in equal pieces, for instance, the ordinal allocation rules of the fair allocation literature (e.g., the Equal Income Walrasian allocation rule), and the ordinal social orderings that we study below do not involve any interpersonal comparisons of utility or welfare. This planner’s ethical preferences revelation principle originates in the fact that the same decision rule can usually be rationalized by several ethical principles. For instance, the market mechanism in an Arrow-Debreu world

---

4 This approach has, however, been challenged by other philosophers (Arneson, 1989; Cohen, 1989) who claim that people should be held responsible only for what is under their control, so that a pure utility deficit that does not derive from individual choice should be deemed a valid claim for a resource transfer. For a criticism, see Fleurbaey (1995).
may be chosen either by a libertarian planner or by a utilitarian planner using appropriate utility functions. It seems clear to us that this does not mean that the libertarian planner makes interpersonal comparisons of utility. It is true that decision rules based on ordinal considerations can be, ex post, rationalized as a utilitarian or any other welfarist rule for an appropriate cardinalization of preferences. But this fact hardly proves that ordinalists are necessarily engaged in (even implicit) utility comparisons.

11.2.2 Social Rankings versus Allocation Rules

A few notations and assumptions will be useful. An economy $e$ is defined by a population $N = \{1, \ldots, n\}$, a profile of characteristics $\theta_N = (\theta_1, \ldots, \theta_n)$, and a set $Z$ of feasible allocations:

$$e = (\theta_N, Z).$$

A typical allocation in $Z$ is denoted $z_N = (z_1, \ldots, z_n)$, where $z_i$ is agent $i$’s bundle. In our applications in this chapter, the only characteristics that describe the agents are their self-centered preferences: $\theta_N = R_N = (R_1, \ldots, R_n)$. For each agent $i$ ($i \in N$), $R_i$ is a complete ordering (with strict preference $P_i$ and indifference $I_i$) over some consumption set $X$ such that $Z \subset X^\mathbb{N}$. We also assume that $X$ is always a subset of a finite-dimensional Euclidean space.

An allocation rule is a correspondence $S$ that selects in each economy of some domain $D^S$ a subset of feasible allocations:

$$S: e \mapsto S(e) \subset Z, \forall e \in D^S.$$

A social ordering function is a function $R$ that defines for each economy of some domain $E^R$ a complete ordering over its feasible allocations:

$$R: e \mapsto R(e) \text{ complete ordering over } Z.$$

Let $\mathcal{R}$ denote the class of all admissible social ordering functions. For a given social ordering $R(e)$, the related strict preference relation will be denoted $P(e)$ and the indifference relation $I(e)$.

In this subsection, we take for granted that the planner may face information, incentive, or observation constraints in such a way that the set of attainable allocations turns out to be a strict subset of $Z$. A first best problem is defined as a problem wherein the planner has the opportunity to make the economy reach any allocation in $Z$. If this is not the case, then the problem is called a second best problem. Let $\mathcal{Y} \subseteq 2^Z$ denote the family of plausible feasible sets in which the planner may have to search for an optimal policy.
In this second best setup, we would like to recall some reasons why social orderings should be preferred to allocation rules.

First, it should be clear that, provided $Z$ is compact, a social ordering function always gives rise to an allocation rule (defined as the selection of allocations socially preferred to all the other allocations). Therefore, if the problem to solve turns out to be a first best problem, then a social ordering function is as useful as an allocation rule.

On the contrary, an allocation rule may prove insufficient to solve all the plausible second best problems in a satisfactory way. There are two reasons for that. The first reason is that some problem $(R_N, Y)$ may not be in the domain of the allocation rule. In other words, the selection may be empty. The second reason is that the selections operated by an allocation rule may be inconsistent in the sense that an allocation selected for a given problem $Y \in \mathcal{Y}$ is not necessarily selected for smaller problems $Y' \subset Y$ containing this allocation.

These two problems, however, can be overcome by imposing nonemptiness and contraction consistency requirements on the allocation rules. But it is well known that these two requirements together are almost equivalent to requiring that the allocation rule be consistent with some social ordering function (provided $\mathcal{Y}$ is sufficiently rich), in the sense that, for any problem, it selects the allocations that are considered as socially better by some complete ranking of the allocations in $Z$.\(^5\)

Therefore, as a social ordering is a natural tool in the second best context, our opinion is that the search for social ordering functions should be the focus of welfare economics, even if allocation rules may be sufficient in the first best context when the available information enables the planner to select any allocation.

11.3 Constructing Social Orderings

Now, we consider that the construction of either a social ordering or an allocation rule should be made on the basis of the properties they satisfy. In other words, it should be axiomatic.\(^6\) In this section, we investigate how to construct social ordering functions in simple division economies. We restrict ourselves to division economies for two reasons. First, we simply need to illustrate the approach, and simple examples are therefore sufficient. Second,

\(^5\) We refer here to Arrow's (1959) proof that condition C4 on a choice function implies that the associated binary relation of social preference is transitive.

\(^6\) There are examples in the literature, however, of nonaxiomatic constructions of allocation rules or social rankings of allocations (e.g., Harsanyi, 1977). For a powerful statement that rules and rankings should be derived axiomatically, see Thomson (2001).
it is a major feature of the axiomatic approach that its conclusions are context dependent; that is, the possibilities to combine axioms vary from one model to another. Consequently, we have to begin our inquiry in a specific model, and the model of division economies is among the best-known models. At the end of this section, we briefly comment on whether the results we obtain can be adapted to other models.

Let the set of feasible allocations be defined with respect to a total amount \( \Omega \in \mathbb{R}_+^l \) of goods and be denoted \( Z(\Omega) \in \mathbb{R}_{ln}^+ \); that is, \( z_N = (z_1, \ldots, z_n) \in Z(\Omega) \Leftrightarrow \sum_{i=1}^{n} z_i \leq \Omega \). Each agent is now equipped with continuous, strictly monotonic, and convex preferences over her consumption set \( \mathbb{R}_+^l \). Let \( \mathcal{R} \) denote the set of all admissible preferences. An economy \( e \) can be described by a list \((R_1, \ldots, R_n, \Omega) \in \mathcal{R}^n \times \mathbb{R}_+^l \). Let \( \mathcal{E} \) denote the class of admissible economies. An allocation \( z_N = (z_1, \ldots, z_n) \) is Pareto efficient for the economy \( e = (R_N, \Omega) \in \mathcal{E} \) if for all \( z_N' = (z_1', \ldots, z_n') \in Z(\Omega), [z'_i I_i z_i, \forall i \in N] \Rightarrow [z'_i I_i z_i, \forall i \in N] \). Let \( P(e) \subset Z(\Omega) \) denote the set of Pareto efficient allocations for \( e \).

Allocation rules and social ordering functions are defined as in the previous section. Their domains are denoted \( \mathcal{E}^S \) and \( \mathcal{E}^R \), respectively.

There are two ways of constructing ordinal social orderings, that is, the direct inquiry and the rationalization of given allocation rules. Since we focus in this chapter on the relationship between the social welfare and the fair allocation approaches, we would like to emphasize the latter way of constructing social ordering functions. It will prove useful, however, to begin with a few remarks on the direct inquiry of social ordering functions.

### 11.3.1 Direct Inquiry

The first way of constructing social orderings is by defining axioms directly bearing on these orderings and trying to combine them. In this subsection, we will give some examples of such axioms, all inspired by the Arrovian axioms.

Let us first define Weak Pareto, Pareto Indifference, and Anonymity, which we consider as essential in our inquiry. Weak Pareto requires that an allocation be deemed socially preferred to another as soon as all the agents strictly prefer the bundle they get in the former to the bundle they get in the latter.

**Weak Pareto:**

\[
\forall e = (R_N, \Omega) \in \mathcal{E}^R, \forall z_N, z'_N \in Z(\Omega),
\quad
[\forall i \in N, z_i P_i z'_i \Rightarrow z_N P(e) z'_N].
\]
Pareto Indifference requires that two allocations considered equivalent by all agents be also considered equivalent by society.

**Pareto Indifference:**

\[ \forall e = (R_N, \Omega) \in \mathcal{E}, \forall z_N, z'_N \in Z(\Omega), \]
\[ [\forall i \in N, z_i I_i z'_i] \Rightarrow z_N I(e) z'_N. \]

Anonymity requires that the name of the agents do not influence the social ranking. The definition of this property requires the following notation. Let \( \pi: N \to N \) denote a permutation of elements of \( N \), and let \( \Pi \) denote the set of all permutations from \( N \) to itself. For any \( n \)-dimensional list \( a_N, \pi(a_N) \) denotes the list obtained by permuting elements of \( a_N \) according to \( \pi \).

**Anonymity:**

\[ \forall e = (R_N, \Omega) \in \mathcal{E}, \forall \pi \in \Pi, \forall z_N, z'_N \in Z(\Omega), \]
\[ z_N R(e) z'_N \iff \pi(z_N) R(\pi(R_N), \Omega) \pi(z'_N). \]

Our three next examples are derived from Arrow’s Independence of Irrelevant Alternatives axiom, in favor of which we would like to argue now. Arrow’s Independence of Irrelevant Alternatives states that the social ranking of two alternatives should be independent of changes in individual preferences, provided these changes do not affect the way in which individuals rank these two allocations (or, more precisely, the bundles they get in these two allocations). This axiom is often justified on grounds of simplicity, but we think that it has an immediate value with respect to the responsibility idea discussed in Section 11.2. If one wants to have the social ordering as independent as possible (under a constraint of compatibility with the Pareto conditions) from individual preferences, such an axiom goes a long way in that direction.

Actually, it goes too far, as the bulk of the social choice literature has uncovered. Therefore, one must look for independence axioms that can be combined with at least the three requirements previously listed. Here are two such axioms. The axiom of Extended Interval Independence requires that the ranking of two allocations remains unaffected by changes in preferences having the double property that the socially preferred allocation increases in

---

7 Arrow’s Independence of Irrelevant Alternatives is also justified on grounds of robustness to changes in the choice set, if one thinks of the derived choice rule. See Arrow (1963).
the individual ranking of all agents, and the other allocation decreases in the
individual ranking of all agents (we have assumed that agents are interested
only in their own consumption, so that by saying that an agent prefers an
allocation to another we mean that she prefers the bundle she is assigned
in the first allocation to the one she is assigned in the second allocation).
Formally,

**Extended Interval Independence:**

\[
\forall e = (R_N, \Omega), e' = (R'_N, \Omega) \in \mathcal{E}, \forall z_N, z'_N \in Z(\Omega),
\]

\[
[\forall i \in N, \forall z \in \mathbb{R}^i_+, z_i R_i z \Rightarrow z_i R'_i z_i \text{ and } z R_i z'_i \Rightarrow z R'_i z'_i] \Rightarrow
\]

\[
[z_N R(e) z'_N \Rightarrow z_N R(e') z'_N].
\]

Examples in the next subsection will show that, in contrast with Arrow’s
Independence of Irrelevant Alternatives, Extended Interval Independence is
easily combined with Weak Pareto, Pareto Indifference, and Anonymity. Let
us observe that this axiom is *not* logically weaker than Arrow’s axiom since a
reversal in an agent’s ranking of the two allocations is allowed. It seems to us
that Extended Interval Independence is one of the most reasonable axioms
that can express the idea that social preferences should be as independent
from individual preferences as possible.

Our second axiom, called Unchanged Contour Independence, is weaker
than Extended Interval Independence, and also weaker than Arrow’s In-
dependence of Irrelevant Alternatives. Unchanged Contour Independence
requires that the ranking of two allocations be independent of changes in
preferences having the property of leaving unaffected the contours (that is,
the upper contour set, the indifference curve, and the lower contour set) of
each agent at the two allocations. Formally,

---

8 An idea that is related to that kind of independence with respect to preferences is to rank
distributions of opportunity sets (see Kranich (1996), Herrero (1996), Herrero, Iturbe, and
Nieto (1998), Bossert, Fleurbaey, and Van de gaer (1999) for examples of such rankings).
In our framework, one might want to rank allocations on the basis of the distributions
of opportunity sets that might have led to them. In economic domains, one may think
in particular of ranking distributions of (not necessarily parallel) budget sets. But given
the fact that individual indirect preferences over budget sets contain as much information
as direct preferences over bundles (see e.g. Blackorby, Primont, and Russell (1978)),
the Pareto conditions obviously require making the social ranking of distributions of budget
sets depend on individual preferences, and then there is little independence from individual
preferences to gain along these lines.

9 We thank M. C. Sanchez for having pointed this fact out to us.
Unchanged Contour Independence:
\[\forall e = (R_N, \Omega), e' = (R'_N, \Omega) \in \mathcal{E}^R, \forall z_N, z'_N \in Z(\Omega),\]
\[\left[\forall i \in N, \forall z \in \mathbb{R}_+^I, z_i I_i z \Rightarrow z_i' I_i' z\right]\]
\[\Rightarrow \left[ z_N R(e) z'_N \Rightarrow z_N R(e') z'_N \right].\]

The last axiom says that a social ordering should only depend on the agents’ preferences, and not on the set of allocations which are feasible. Actually, such a requirement is implicit in the Arrovian framework when the set of alternatives is viewed as the largest possible set of conceivable social states, whereas the social choice may take place among elements of a strict subset of this large set.

Independence of the Feasible Set:
\[\forall e = (R_N, \Omega), e' = (R_N, \Omega') \in \mathcal{E}, \forall z_N, z'_N \in Z(\Omega) \cap Z(\Omega'),\]
\[z_N R(e) z'_N \Leftrightarrow z_N R(e') z'_N.\]

We have restricted ourselves to axioms inspired by the Arrovian social choice theory. We believe though that economic domains give us the possibility to express more specific ethical principles. We do not develop this line of research here. But to conclude, we simply say that the analysis of social ordering functions in economic domains by direct axiomatic inquiry seems to us an urgent task.

### 11.3.2 Rationalizing Allocation Rules

In this subsection, we study how social ordering functions can be constructed when desirable allocation rules have been identified, and one would like to extend them into fine-grained social orderings. In other words, given an allocation rule \( S \) defined on \( \mathcal{E}^S \), we look for a social ordering function \( R^S \in \mathcal{R} \) that rationalizes it in the sense that the allocations in \( S(e) \) must always be top ranked by the social ordering \( R^S(e) \).

**Rationalization of \( S \):**
\[\forall e \in \mathcal{E}^S,\]
\[S(e) = \{ z_N \in Z(\Omega) \mid \forall z'_N \in Z(\Omega), z_N R^S(e) z'_N \}.\]
Clearly, an allocation rule can always be rationalized by a two class social ordering function where selected allocations are socially indifferent to each other and form the first class, and nonselected allocations are also socially indifferent to each other and form the second class. But the resulting social orderings typically fail to satisfy even the three basic properties discussed at the beginning of the previous subsection (Weak Pareto, Pareto Indifference, and Anonymity).

Theorem 11.1 proves that the three major solutions to the fair division problem (see Moulin, 1990) can be rationalized by social ordering functions satisfying the three basic properties as well as Extended Interval Independence, and, in one of the three cases, Independence of the Feasible Set. These allocation rules are the Fixed Numeraire Egalitarian Equivalent rule, the Pazner-Schmeidler Egalitarian Equivalent rule, and the Equal Income Walrasian rule.

The Fixed Numeraire $z^*$ Egalitarian equivalent rule selects all the Pareto efficient allocations that have the property that each agent is indifferent between her bundle and a reference bundle defined as a multiple of the numeraire $z^*$.

The Fixed Numeraire $z^*$ Egalitarian Equivalent rule $E_{z^*}$:

$$\forall e = (R_N, \Omega) \in \mathcal{E}, z_N = (z_1, \ldots, z_n) \in P(e),$$

$$z_N \in E_{z^*}(e) \iff \exists \lambda \in \mathbb{R}_+ \text{ s.t. } z_i \lambda z^*, \forall i \in N.$$ 

The second allocation rule is the Pazner-Schmeidler Egalitarian Equivalent rule $E_{\Omega}$. Rather than taking a fixed numeraire $z^*$, it takes the reference bundle proportional to $\Omega$.

The Pazner-Schmeidler Egalitarian Equivalent rule $E_{\Omega}$:

$$\forall e = (R_N, \Omega) \in \mathcal{E}, z_N = (z_1, \ldots, z_n) \in P(e),$$

$$z_N \in E_{\Omega}(e) \iff \exists \lambda \in \mathbb{R}_+ \text{ s.t. } z_i \lambda \Omega, \forall i \in N.$$ 

The third rule is the Equal Income Walrasian rule $W$.

\[^{10}\text{We disregard the Strong Pareto condition in this chapter only for simplicity. Additional results relative to this condition are readily obtained, and are left here to the reader.}\]
Utilitarianism versus Fairness

Equal Income Walrasian rule W

\[ \forall e = (R_N, \Omega) \in E, z_N \in P(e), \]
\[ [z_N \in W(e)] \iff [\exists p \in \mathbb{R}^l_+, \forall i \in N, \forall x \in \mathbb{R}^l_+, px \leq \frac{\omega}{n} \implies z_i R_i x]. \]

Theorem 11.1:

1. The Fixed Numeraire \( z^* \) Egalitarian Equivalent rule can be rationalized by a social ordering function satisfying Weak Pareto, Pareto Indifference, Anonymity, Extended Interval Independence, and Independence of the Feasible Set.
2. The Pazner-Schmeidler Egalitarian Equivalent rule \( E_{\Omega^1} \) and the Equal Income Walrasian rule \( W \) can be rationalized by a social ordering function satisfying Weak Pareto, Pareto Indifference, Anonymity, and Extended Interval Independence.
3. However, there is no social ordering function satisfying Weak Pareto and Independence of the Feasible Set that rationalizes either \( E_{\Omega^1} \) or \( W \).

We view this theorem as the success evidence of the approach. It tells us not only that our alternative to Arrow’s Independence of Irrelevant Alternatives can be combined with Weak Pareto, Pareto Indifference, and Anonymity, but it tells us also that in addition to satisfying these properties, the social ordering functions we obtain may rationalize famous equitable allocation rules. We also have to emphasize the negative content of point 3: A social ordering function rationalizing an equitable allocation rule is quite likely to depend on the resources of the economy.\(^{11}\)

We begin the proof of the theorem here by defining social ordering functions that rationalize the three solutions and that satisfy the properties stated in Theorem 11.1. The proof is completed in the appendix.

A social ordering function \( R^{E_{\Omega^1}} \) rationalizing \( E_{\Omega^1} \) can be constructed by cardinalizing preferences as follows: \( u^\ast_i(z_i) = \lambda \iff z_i R_i \lambda z^* \), for all \( i \in N \). Then, we define \( R^{E_{\Omega^1}}(e) \) by applying the maximin criterion to the vector \( (u^\ast_i(z_i); i \in N) \).

\(^{11}\) Eisenberg (1961) and Milleron (1970) have proved that the impossibility of rationalizing \( W \) by a social ordering function satisfying Independence of the Feasible Set disappears on the subdomain of homothetic preferences. The social ordering they propose is representable, interestingly, by the product of values taken by homogeneous (i.e., least concave) representations of the agents’ preferences. We do not know of any result of this sort for the \( E_{\Omega^1} \) rule.
The Social Ordering Function $R_{E_{\omega}}$: 

$\forall e = (R_N, \Omega) \in \mathcal{E}, z_N = (z_1, \ldots, z_n), z'_N = (z'_1, \ldots, z'_n) \in \mathbb{R}_+^{ln}$,

$z_N R_{E_{\omega}}(e) z'_N \iff \min_{i \in N} \{u_i^\omega(z_i)\} \leq \min_{i \in N} \{u_i^\omega(z'_i)\}$.

A social ordering function $R_{E_{\omega}}$ rationalizing $E_{\omega}$ can be constructed by cardinalizing preferences $R_i$ as follows: $u_i^\omega(z_i) = \lambda \iff z_i I_i \lambda \omega$, for all $i \in N$. Then, we define $R_{E_{\omega}}(e)$ by applying the maximin criterion to the vector $(u_i^\omega(z_i); i \in N)$.

The Social Ordering Function $R_{W}$:

$\forall e = (R_N, \Omega) \in \mathcal{E}, z_N = (z_1, \ldots, z_n), z'_N = (z'_1, \ldots, z'_n) \in \mathbb{R}_+^{ln}$,

$z_N R_{W}(e) z'_N \iff \min_{i \in N} \{u_i^\omega(z_i)\} \leq \min_{i \in N} \{u_i^\omega(z'_i)\}$.

Theorem 11.1 does not allow us, however, to identify the class of allocation rules that can be rationalized by social ordering functions satisfying the three basic properties as well as Extended Interval Independence or Independence of the Feasible Set. We still do not have general results regarding this question.

We find interesting, however, to complete this section by defining two ways of constructing social ordering functions from allocation rules. First, we introduce the appropriate concept. We propose to call a social ordering functional a function associating to each allocation rule in some admissible domain a social ordering function that rationalizes it. The two social
ordering functionals are both based on a real-valued function computing what could be called the value of an allocation, depending on the parameters of the economy. Then, the social ordering function is derived from the principle that an allocation is socially preferred if its value is greater.

Let us consider a first way of computing the value of an allocation. Let $S$ denote the allocation rule that one tries to rationalize. The value of an allocation $z_N$ for an economy $e = (R_N, \Omega)$ is given by the highest share $\lambda$ satisfying the property that an $S$-optimal allocation $z'_N$ exists in the economy $(R_N, \lambda \Omega)$ to which all agents weakly prefer $z_N$. Formally, $v: \mathbb{R}^n_+ \times E^S \to \mathbb{R}_+$ is defined by: for all $e = (R_N, \Omega) \in E^S$,

$$v(z_N, R_N, \Omega) = \sup \{ \lambda | \exists z'_N \in S(R_N, \lambda \Omega) \text{ s.t. } z_i R_i z'_i, \forall i \in N \}.$$ 

We construct $R^S$ as follows: $z_N R^S(e) z'_N \iff v(z_N, e) \geq v(z'_N, e)$.

The second value function is similar to the first one, except that the supremum is no longer defined with respect to economies $(R_N, \lambda \Omega)$ but to economies $(R'_N, \lambda \Omega)$, where the choice of $R'_N$ is simply restricted by the condition that agent $i$’s indifference curve through $z_i$ be the same at $R_i$ as at $R'_i$ (we do not define this value function formally, to avoid long mathematical notations).

These value functions are clearly inspired by the Debreu (1951) coefficient of resource utilization. Actually, Debreu’s coefficient corresponds to the function we obtain by applying either social ordering functional to the Pareto rule. However, the careful reader will have noted that by applying our first functional to $E^*_\omega$ or to $E_\Omega$, we come to the corresponding social ordering functions $R^{E^*_\omega}$ or $R^{E_\Omega}$, whereas the second functional, applied to $W$, gives us $R^W$. We think that social ordering functionals like these ones could prove extremely useful in the general research for appealing social ordering functions. Finally, let us note that any social ordering function obtained by applying the second functional satisfies Extended Interval Independence.

Before closing this subsection, we note that the results presented here in the context of division economies can be adapted to private or public good production economies. For instance, in the provision of one public good model, the solution discussed by Moulin (1987) can be rationalized in a similar way as $E^*_\omega$, whereas the Lindhal rule can be rationalized in a similar way as $W$. In conclusion, even in the absence of a clear general result, it does not seem too difficult to get allocation rules rationalizable by social ordering functions satisfying Weak Pareto, Pareto Independence, Anonymity, and Extended Interval Independence.
11.4 Concluding Comments

This chapter is exploratory and provides more questions than solutions. However, we hope that the kind of social orderings proposed here can contribute to bridging the gap between the social welfare approach and the fair allocation approach. We hope, above all, that such social ordering function will come closer to policy applications than traditional social welfare functions or first best allocation rules. The application of such orderings to second best problems in public economics might yield new policy recommendations that would rely only on ordinal noncomparable information.

Many questions remain open. First, the list of axioms that can bear on social orderings is far from closed. Second, the list of social ordering functionals must also be enlarged. Third, the characterization of allocation rules that can be rationalized by social ordering functions satisfying appealing properties is still an open problem. The axioms defined in this chapter and Theorem 11.1, however, already show that the analysis of social ordering functions is not limited at uncovering impossibility results.

References


Utilitarianism versus Fairness


APPENDIX

Proof of Theorem 1: The straightforward proof that $R^{E_0}$, $R^{E_{r'}}$, and $R^W$ all satisfy Weak Pareto, Pareto Indifference, and Anonymity is left to the reader.

Let us consider Extended Interval Independence. Let $i$ and $z_i$ be fixed. Let $u_i^r$, $u_i^{r'}: \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ representing $R_i$ and $R_i'$, respectively, be defined as in the construction of $R^{E_{r'}}$. If for all $z \in \mathbb{R}_+^n$, $z; R_i z \Rightarrow z; R_i' z$, then $u_i^r(z_i) \leq u_i^{r'}(z_i)$, whereas if for all $z \in \mathbb{R}_+^n$, $z; R_i z_i \Rightarrow z; R_i' z_i$, then $u_i^r(z_i) \geq u_i^{r'}(z_i)$. The claim that $R^{E_{r'}}$ satisfies Extended Interval Independence follows directly from this fact.

By replacing $u_i^{r'}$ by $u_i^{r''}$, we can prove that $R^{E_0}$ satisfies Extended Interval Independence.

Let $R_N, R'_N \in \mathcal{R}^n$ be given with the property that for all $i \in N$, for all $z \in \mathbb{R}_+^n$, $z; R_i z \Rightarrow z; R_i' z$. Then we have $\text{co}(\bigcup_{i \in N} U(z_i, R_i)) \supseteq \text{co}(\bigcup_{i \in N} U(z_i, R'_i))$, and the proof that $R^W$ satisfies Extended Interval Independence is straightforward.
Let us now consider Independence of the Feasible Set. First, it is clear that the \( u_i^* \) functions do not depend on \( \Omega \) at all and, consequently, \( R_{E,<} \) satisfies Independence of the Feasible Set.

Finally, take economies with \( n = l = 2 \), and preferences represented by \( U_1(x, y) = 10\sqrt{x} + y \), \( U_2(x, y) = x + 10\sqrt{y} \). Consider \( W \) first. Quasi-linearity and strict convexity of the preferences implies that the Walrasian equilibrium allocation is always unique. Let \( \gamma \) denote the ratio of 1’s income over 2’s income. Then for \( e \) with \( W = (25, 50) \), \( W(e) = z^1_N = ((19.16, 17.39), (5.84, 32.61)) \) with \( U_1 = 61.17 \), \( U_2 = 62.94 \), and if \( \gamma = 1.095 \), one has the allocation \( z^2_N = ((19.85, 18.52), (5.15, 31.48)) \) with \( U_1 = 63.08 \), \( U_2 = 61.26 \). And we have symmetric values for \( e' \) with \( W' = (50, 25) \) (allocations \( z^3_N \) and \( z^4_N \)). We see that \( z^1_N P(e)z^2_N \) and \( z^3_N P(e')z^4_N \) by rationalization of \( W \) and Independence of the Feasible Set. Let \( e'' \) be the economy with \( W'' = (50, 50) \). By Independence of the Feasible Set, \( R(e'') \) coincides with \( R(e) \) in \( e \), and with \( R(e') \) in \( e' \). Besides, \( z^1_N P(e'')z^3_N \) and \( z^4_N P(e'')z^1_N \) by Weak Pareto. Therefore, \( z^1_N P(e'')z^3_N P(e'')z^4_N P(e'')z^1_N \).

Now consider \( E_{\Omega} \). We have \( E_{\Omega}(e) = z^1_N = ((19.11, 17.30), (5.89, 32.70)) \) with \( U_1 = 61.01 \), \( U_2 = 63.08 \), and \( z^2_N = ((19.89, 18.57), (5.11, 31.43)) \) \( \in P(e) \) with \( U_1 = 63.16 \) and \( U_2 = 61.18 \). The rest of the example is constructed as in the preceding paragraph. Q.E.D.